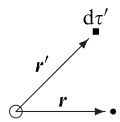
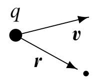
7.3 Electromagnetic fields (general)

Field relationships

Conservation of charge	$\nabla\cdot\boldsymbol{J}=-\dfrac{\partial\rho}{\partial t}$	(7.39)	$\boldsymbol{J}$ current density; ρ charge density; t time
Magnetic vector potential	$\boldsymbol{B}=\nabla\times\boldsymbol{A}$	(7.40)	$\boldsymbol{A}$ vector potential
Electric field from potentials	$\boldsymbol{E}=-\dfrac{\partial\boldsymbol{A}}{\partial t}-\nabla\phi$	(7.41)	ϕ electrical potential
Coulomb gauge condition	$\nabla\cdot\boldsymbol{A}=0$	(7.42)	
Lorenz gauge condition	$\nabla\cdot\boldsymbol{A}+\dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t}=0$	(7.43)	c speed of light
Potential field equations[a]	$\dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}-\nabla^2\phi=\dfrac{\rho}{\epsilon_0}$	(7.44)	
	$\dfrac{1}{c^2}\dfrac{\partial^2\boldsymbol{A}}{\partial t^2}-\nabla^2\boldsymbol{A}=\mu_0\boldsymbol{J}$	(7.45)	
Expression for ϕ in terms of ρ[a]	$\phi(\boldsymbol{r},t)=\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{volume}}\dfrac{\rho(\boldsymbol{r}',t-\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert/c)}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert}\,\mathrm{d}\tau'$	(7.46)	$\mathrm{d}\tau'$ volume element; $\boldsymbol{r}'$ position vector of $\mathrm{d}\tau'$
Expression for $\boldsymbol{A}$ in terms of $\boldsymbol{J}$[a]	$\boldsymbol{A}(\boldsymbol{r},t)=\dfrac{\mu_0}{4\pi}\displaystyle\int_{\text{volume}}\dfrac{\boldsymbol{J}(\boldsymbol{r}',t-\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert/c)}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert}\,\mathrm{d}\tau'$	(7.47)	μ_0 permeability of free space

[a]Assumes the Lorenz gauge.

Liénard–Wiechert potentials[a]

Electrical potential of a moving point charge	$\phi=\dfrac{q}{4\pi\epsilon_0(\lvert\boldsymbol{r}\rvert-\boldsymbol{v}\cdot\boldsymbol{r}/c)}$	(7.48)	q charge; $\boldsymbol{r}$ vector from charge to point of observation; $\boldsymbol{v}$ particle velocity
Magnetic vector potential of a moving point charge	$\boldsymbol{A}=\dfrac{\mu_0 q\boldsymbol{v}}{4\pi(\lvert\boldsymbol{r}\rvert-\boldsymbol{v}\cdot\boldsymbol{r}/c)}$	(7.49)	

[a]In free space. The right-hand sides of these equations are evaluated at retarded times, i.e., at $t'=t-\lvert\boldsymbol{r}'\rvert/c$, where $\boldsymbol{r}'$ is the vector from the charge to the observation point at time t'.

Maxwell's equations

Differential form:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad (7.50)$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad (7.52)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (7.54)$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \qquad (7.56)$$

Integral form:

$$\oint_{\text{closed surface}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \frac{1}{\epsilon_0} \int_{\text{volume}} \rho \, \mathrm{d}\tau \qquad (7.51)$$

$$\oint_{\text{closed surface}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = 0 \qquad (7.53)$$

$$\oint_{\text{loop}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \qquad (7.55)$$

$$\oint_{\text{loop}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I + \mu_0 \epsilon_0 \int_{\text{surface}} \frac{\partial \boldsymbol{E}}{\partial t} \cdot \mathrm{d}\boldsymbol{s} \qquad (7.57)$$

Equation (7.51) is "Gauss's law"

Equation (7.55) is "Faraday's law"

$\boldsymbol{E}$	electric field	$\mathrm{d}\boldsymbol{s}$	surface element
$\boldsymbol{B}$	magnetic flux density	$\mathrm{d}\tau$	volume element
$\boldsymbol{J}$	current density	$\mathrm{d}\boldsymbol{l}$	line element
ρ	charge density	Φ	linked magnetic flux ($=\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s}$)
		I	linked current ($=\int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{s}$)
		t	time

Maxwell's equations (using D and H)

Differential form:

$$\nabla \cdot \boldsymbol{D} = \rho_{\text{free}} \qquad (7.58)$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad (7.60)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (7.62)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad (7.64)$$

Integral form:

$$\oint_{\text{closed surface}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{s} = \int_{\text{volume}} \rho_{\text{free}} \, \mathrm{d}\tau \qquad (7.59)$$

$$\oint_{\text{closed surface}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = 0 \qquad (7.61)$$

$$\oint_{\text{loop}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \qquad (7.63)$$

$$\oint_{\text{loop}} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I_{\text{free}} + \int_{\text{surface}} \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{s} \qquad (7.65)$$

$\boldsymbol{D}$	displacement field	$\boldsymbol{E}$	electric field
ρ_{free}	free charge density (in the sense of $\rho = \rho_{\text{induced}} + \rho_{\text{free}}$)	$\mathrm{d}\boldsymbol{s}$	surface element
$\boldsymbol{B}$	magnetic flux density	$\mathrm{d}\tau$	volume element
$\boldsymbol{H}$	magnetic field strength	$\mathrm{d}\boldsymbol{l}$	line element
$\boldsymbol{J}_{\text{free}}$	free current density (in the sense of $\boldsymbol{J} = \boldsymbol{J}_{\text{induced}} + \boldsymbol{J}_{\text{free}}$)	Φ	linked magnetic flux ($=\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s}$)
		I_{free}	linked free current ($=\int \boldsymbol{J}_{\text{free}} \cdot \mathrm{d}\boldsymbol{s}$)
		t	time

Relativistic electrodynamics

Lorentz transformation of electric and magnetic fields	$\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel$	(7.66)	$\boldsymbol{E}$ electric field
	$\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})_\perp$	(7.67)	$\boldsymbol{B}$ magnetic flux density
	$\boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel$	(7.68)	$'$ measured in frame moving at relative velocity $\boldsymbol{v}$
	$\boldsymbol{B}'_\perp = \gamma(\boldsymbol{B}-\boldsymbol{v}\times\boldsymbol{E}/c^2)_\perp$	(7.69)	γ Lorentz factor $=[1-(v/c)^2]^{-1/2}$
			$\parallel$ parallel to $\boldsymbol{v}$
			$\perp$ perpendicular to $\boldsymbol{v}$
Lorentz transformation of current and charge densities	$\rho' = \gamma(\rho - vJ_\parallel/c^2)$	(7.70)	$\boldsymbol{J}$ current density
	$J'_\perp = J_\perp$	(7.71)	ρ charge density
	$J'_\parallel = \gamma(J_\parallel - v\rho)$	(7.72)	
Lorentz transformation of potential fields	$\phi' = \gamma(\phi - vA_\parallel)$	(7.73)	ϕ electric potential
	$A'_\perp = A_\perp$	(7.74)	$\boldsymbol{A}$ magnetic vector potential
	$A'_\parallel = \gamma(A_\parallel - v\phi/c^2)$	(7.75)	
Four-vector fields[a]	$\underset{\sim}{\boldsymbol{J}} = (\rho c, \boldsymbol{J})$	(7.76)	$\underset{\sim}{\boldsymbol{J}}$ current density four-vector
	$\underset{\sim}{\boldsymbol{A}} = \left(\frac{\phi}{c}, \boldsymbol{A}\right)$	(7.77)	$\underset{\sim}{\boldsymbol{A}}$ potential four-vector
	$\Box^2 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}, -\nabla^2\right)$	(7.78)	$\Box^2$ D'Alembertian operator
	$\Box^2 \underset{\sim}{\boldsymbol{A}} = \mu_0 \underset{\sim}{\boldsymbol{J}}$	(7.79)	

[a]Other sign conventions are common here. See page 65 for a general definition of four-vectors.